# GENERALIZATIONS OF THE FLOOR AND CEILING FUNCTIONS USING THE STERN-BROCOT TREE

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Blekinge Institute of Technology Research report No. 2006:02

# Generalizations of the floor and ceiling functions using the Stern-Brocot tree

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#### Abstract

We consider a fundamental number theoretic problem where practial applications abound. We decompose any rational number  $\frac{a}{b}$  in c ratios as evenly as possible while maintaining the sum of numerators and the sum of denominators. The minimum  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and maximum  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  of the ratios give rational estimates of  $\frac{a}{b}$  from below and from above. The case c = b gives the usual floor and ceiling functions. We furthermore define the difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ , which is zero iff  $c \leq GCD(a, b)$ , quantifying the distance to relative primality.

A main tool for investigating the properties of  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ ,  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  is the Stern-Brocot tree, where all positive rational numbers occur in lowest terms and in size order. We prove basic properties such that there is a unique decomposition that gives both  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ . It turns out that this decomposition contains at most three distinct ratios.

The problem has arisen in a generalization of the 4/3-conjecture in computer science.

**Keywords**: Floor function, ceiling function, mediant, relative primality, Stern-Brocot tree.

# 1 Introduction

In this paper we study optimal ways to decompose a rational number  $\frac{a}{b}$  in c ratios, while preserving the sum of numerators and the sum of denominators. This is done so that all ratios are as close as possible to  $\frac{a}{b}$ . We are interested in the minimum, maximum of the ratios, and of the difference of these two numbers. The problem is a fundamental number theoretic problem, and has very practical implications. However, the problem has arisen in a computer science context. The main results in the present paper are important in the companion paper [7], which otherwise is independent. In that paper a general version of the 4/3-conjecture, well known in computer science, is solved.

The present paper is purely mathematical. Here we take advantage of the Stern-Brocot tree to prove the existence of optimal decompositions in c ratios, and to find these decompositions.

Given three integers a, b and c where  $1 \le c \le b$ , we consider sets of c quotients  $\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}$  so that  $a = \sum_{i=1}^{c} a_i$  and  $b = \sum_{i=1}^{c} b_i$ . Here all  $a_i$  are integers and all  $b_i$  are positive integers, i.e.  $1 \le b_i \le b$  for all i. Such a set is called a *c*-decomposition of  $\frac{a}{b}$ . Pick a *c*-decomposition where  $\min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$  is maximal. For such a decomposition we denote  $\lfloor \frac{a}{b} \rfloor_c = \min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$ . We call  $\lfloor \frac{a}{b} \rfloor_c$  the *c*-floor ratio of a and b. This term is motivated by the fact that we have  $\lfloor \frac{a}{b} \rfloor_1 = \frac{a}{b}$ and  $\begin{bmatrix} a \\ b \end{bmatrix}_b = \begin{bmatrix} a \\ b \end{bmatrix}$ , so the quantity generalizes the floor function. We refrain from

writing  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  with a fraction bar, as  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ , since  $\begin{bmatrix} da \\ db \end{bmatrix}_c \neq \begin{bmatrix} a \\ b \end{bmatrix}_c$  in general. We may similarly define a generalized ceiling function. For a decomposition  $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \dots, \begin{bmatrix} a_c \\ b_c \end{bmatrix}$  where  $\max(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$  is minimal, we denote  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \max(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$ , which is the *c*-ceiling ratio of *a* and *b*. We prove that there is a decomposition folling both the minimum and the maximum in the second s fulfilling both the minimum and the maximum, i.e. where  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \min(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c})$ and  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \max(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c})$  (Lemma 18). We furthermore define the *ceiling-floor difference* as

$$\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_c - \begin{bmatrix} a \\ b \end{bmatrix}_c$$

We have  $\begin{bmatrix} a \\ b \end{bmatrix}_c = 0$  if and only if  $c \leq GCD(a, b)$ , since only if  $c \leq GCD(a, b)$ there are decompositions so that all ratios are equal. If a is not a multiple of b the difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  increases from 0 to 1 when c increases from 1 to b. The quantity  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  quantifies the distance to divisibility of a by b, where c can be seen as a crudeness parameter. This is reflected in the property  $\begin{bmatrix} da \\ db \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_{\lceil c/d\rceil}$  (Lemma 11). The difference  $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]_c$  has practical interpretations. One is the "unavoidable unfairness" if a objects are shared among b persons who are subdivided in cgroups (see Section 4).

The sequences of floor or ceiling ratios may be denoted without index, i.e.

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left( \begin{bmatrix} a \\ b \end{bmatrix}_1, ..., \begin{bmatrix} a \\ b \end{bmatrix}_b \right), \text{ and}$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \left( \begin{bmatrix} a \\ b \end{bmatrix}_1, ..., \begin{bmatrix} a \\ b \end{bmatrix}_b \right).$$

The sequence  $\begin{vmatrix} a \\ b \end{vmatrix}$  is decreasing from  $\frac{a}{b}$  to  $\begin{vmatrix} a \\ b \end{vmatrix}$  as a function of c, while  $\begin{bmatrix} a \\ b \end{bmatrix}$  is increasing from  $\frac{a}{b}$  to  $\left\lceil \frac{a}{b} \right\rceil$ . We here use the terms "increasing" and "decreasing" in the forms that allow equality – e.g. f(x) is increasing if  $f(x) \leq f(y)$  for all x < y.

The parameter c specifies the number of ratios in which to divide  $\frac{a}{b}$ , slightly similarly to how the denominator b in  $\frac{a}{b}$  specifies the number of parts in which to divide a. The notion of c as an "extra denominator" of a special kind is supported by that the abbreviation formula  $\frac{da}{db} = \frac{a}{b}$  can be generalized to also include c, since we have

$$\begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}, \quad \begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}, \text{ and } \begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}$$

for any positive integer d (Lemma 12). It is however more to the point to describe c as the degree of crudeness in how we estimate  $\frac{a}{b}$ , alternatively to regard b - c + 1 as the degree of accuracy. This is natural since c = 1 give maximal accuracy,  $\begin{bmatrix} a \\ b \end{bmatrix}_1 = \begin{bmatrix} a \\ b \end{bmatrix}_1 = \frac{a}{b}$ , while it is minimal for c = b where we get the floor and ceiling functions  $\begin{bmatrix} a \\ b \end{bmatrix}_b = \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ . In this paper we give basic properties of  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and show how the c-

In this paper we give basic properties of  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and show how the *c*-floor and *c*-ceiling ratios effectively may be calculated by using the Stern-Brocot tree. The main reference for the Stern-Brocot tree is [4]. In this tree all positive rational numbers are generated exactly once, and all occur in shortest terms. The link between the *c*-floor and *c*-ceiling ratios and the Stern-Brocot tree is provided by the operation

$$\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2},$$

here denoted by  $\oplus$ . The number  $\frac{a_1+a_2}{b_1+b_2}$  is called the *mediant* of  $\frac{a_1}{b_1}$  and  $\frac{a_2}{b_2}$ . It is the main operation of construction of the Stern-Brocot tree, and expresses that the sums of numerators and denominators are preserved in a decomposition. In Section 4 applications in number theory and discrete linear algebra are described briefly.

The operation  $\oplus$  has natural practical applications. Consider a situation where we have  $a_1$  kg of a certain gas in a container of volume  $b_1$  litres, with density  $a_1/b_1$ , and similarly for  $a_2$  kg of a certain gas in a neighbouring container of volume  $b_2$  litres. If the containers are merged, for example by removing a wall between the containers, we get the density  $(a_1 + a_2)/(b_2 + b_2)$  in the larger merged volume. This instance has obvious discrete counterparts if  $a_1, a_2, b_2$  and  $b_2$  are all integers.

In this paper we are interested in the inverse mediant operation, meaning that we will go backwards in the Stern-Brocot tree. Given two numbers a and b, we want to find a decomposition with  $\frac{a_1}{b_1} \oplus ... \oplus \frac{a_c}{b_c} = \frac{a}{b}$ . We are interested in a decomposition that is *uniform*, i.e., the ratios  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}$  should be as equal as possible. This problem is trivial for continuous sets of numbers, in which case all ratios can be taken to be equal. It is not trivial if  $a_i \in \mathbb{Z}$  and  $b_i \in \mathbb{Z}_+$  for all i, and the difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_c - \begin{bmatrix} a \\ b \end{bmatrix}_c$  quantifies the distance to an even distribution. For discrete sets we thus define:

**Definition 1** Assume that  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}_+$ . A decomposition  $\frac{a}{b} = \frac{a_1}{b_1} \oplus \dots \oplus \frac{a_c}{b_c}$  is uniform from below if there is no other decomposition with larger  $\min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$ . Similarly, it is uniform from above if there is no other decomposition with smaller  $\max(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c})$ . It is uniform if it has both properties.

It turns out that for any  $\frac{a}{b}$  there exist a uniform partition. Furthermore it is unique and contains at most three distinct ratios (Lemma 18). The Stern-Brocot tree also provides a fast algorithm to calculate the numbers  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ .

The paper is organized as follows. In Section 2 we present basic properties of the mediant and of the Stern-Brocot tree, concluding with previous research. In Section 3 we present and prove basic properties of the c-floor and the c-ceiling ratios, and the ceiling-floor difference. In Section 4 a few applications of these functions are discussed.

# 2 The mediant and the Stern-Brocot tree

## 2.1 Previous research

## 2.1.1 Number theory and the Stern-Brocot graph

Moritz Abraham Stern (1807–1894) succeeded Carl Friedrich Gauss in Göttingen. In 1858 he published the article *Über eine zahlentheoretische Funktion* [8], which contains the first publication of a tree which later came to be known as the Stern-Brocot tree. Independently, the clockmaker Achille Brocot 1861 presented the same tree in a paper about efficient use of systems of cogwheels [2]. Thus, from the very beginning the number theoretic content of the tree was accompagnied by applications, similarly to how this paper has emerged from problems in the computer science companion paper [7] (see Section 4). The Stern-Brocot tree was reintroduced by R. Graham, D. Knuth and O. Patasnik in [4], and has since then been the subject of research. For example, in [6] M. Niqui devises algoritms for exact algorithms for rational and real numbers based on the Stern-Brocot tree.

#### 2.1.2 Computer science

Computer science problems are often very close to pure combinatorial or number theoretic problems. In a well-known binpacking problem we have n positive numbers  $\mathbf{x} = (x_1, ..., x_n)$  and want to find a partition A of these numbers in ksets  $A_1, ..., A_k$ , k < n, so that  $f(A, \mathbf{x}) = \max_{1 \le j \le k} (\sum_{i \in A_j} x_i)$  is minimal. We may denote this minimum by  $\tilde{f}(\mathbf{x}) = \min_A f(A, \mathbf{x})$ 

In the 4/3-conjecture two cases of binpacking are compared in the case k = 2. Normal binpacking, as above, is compared to a counterpart with an extra liberty during the packing. Here one of the numbers  $x_i$  may be split in two positive numbers  $x_{i,1}$  and  $x_{i,2}$  whose sum is  $x_i$ . We denote by  $f'(A, \mathbf{x}) = \min_{\text{split a } x_i} (\max_{1 \le j \le k} (\sum_{i \in A_j} x_i))$  and  $\tilde{f}'(\mathbf{x}) = \min_A f'(A, \mathbf{x})$ . It is trivial that  $\min_{\mathbf{x}} (f/f') = 1$ . The 4/3-conjecture states that  $\max_{\mathbf{x}} (f/f') = 4/3$ . This statement was conjectured by Liu 1972 [5] and proved by Coffman and Garey 1993 [3], all in a computer science context.

In computer science context, a set of numbers  $(x_1, ..., x_n)$  may represent a parallel program, the k partition sets correspond to the processors of a multiprocessor with k processors, a partition A is a schedule of the parallel program, and a split of a program from  $x_i$  into  $x_{i,1}$  and  $x_{i,2}$ , where  $x_i = x_{i,1} + x_{i,2}$ , is called a preemption. A partition where splits are allowed is then called a preemptive schedule. Braun and Schmidt proved 2003 a formula that compares preemptive schedules with i preemptions to a schedule with unlimited number of preemptions in the worst case, using a multiprocessor with m processors [1]. The comparison is made in terms of the ratio of completion times for a program that maximizes this ratio, when assuming optimal schedules in both cases. They show that no more than m-1 preemptions are needed in the unlimited case. They generalized the bound 4/3 to the formula 2 - 2/(m/(i+1) + 1), which also may be written as 2m/(m+i+1).

The paper [7] generalizes the problem considered by Braun and Schmidt into an optimal comparison of i preemptions to j preemptions, using a multiprocessor with m processors. In the case  $j \leq m - i - 1$  we find the optimal bound

$$2\frac{\lfloor j/(i+1)\rfloor+1}{\lfloor j/(i+1)\rfloor+2}$$

It turns out that in the case  $j \leq m - i - 1$ , the floor ratio provides an explicit formula:  $2 \begin{bmatrix} i+j+1\\2i+j+2 \end{bmatrix}_{\min(m,i+j+1)-j}$ . This problem was a source for the present paper.

#### 2.2The mediant – basic properties

In this section we consider properties of the mediant operation  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1+a_2}{b_1+b_2}$ 

In this section we consider properties of the mediant operation  $b_1 \\end{black} b_2 \\b_1+b_2$ First we formulate immediate properties. It is obvious that  $\frac{a_1}{b_1} < \frac{a_1+a_2}{b_1+b_2} < \frac{a_2}{b_2}$  if  $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ , unless  $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ , in which case  $\frac{a_1}{b_1} = \frac{a_1+a_2}{b_1+b_2} = \frac{a_2}{b_2}$ . The strict inequalities are important for the Stern-Brocot tree. The mediant operation is associative,  $(\frac{a_1}{b_1} \oplus \frac{a_2}{b_2}) \oplus \frac{a_3}{b_3} = \frac{a_1}{b_1} \oplus (\frac{a_2}{b_2} \oplus \frac{a_3}{b_3})$ , so we may simply write  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} \oplus \frac{a_3}{b_3}$ , and commutative  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} \oplus \frac{a_3}{b_1} \oplus \frac{a_1}{b_1}$ . We will also need the rule  $\frac{a_1+db_1}{b_1} \oplus \frac{a_2+db_2}{b_2} \oplus \dots \oplus \frac{a_c+db_c}{b_2} = \frac{a_1}{b_1} \oplus \frac{a_2}{b_2} \dots \oplus \frac{a_c}{b_c} + d$ . We assign higher priority to  $\oplus$  than +, so that  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} + d$  is to be read  $(\frac{a_1}{b_1} \oplus \frac{a_2}{b_2}) + d$ . The mediant can be regarded as a weighted mean value. The quantity  $w_1x_1 + w_1 + w_1 + w_2 + w_1 + w_1 + w_1 + w_1 + w_1 + w_2 + w_1 + w_1$ 

The mediant can be regarded as a weighted mean value. The quantity  $w_1x_1 +$  $w_2x_2$  is the arithmetic weighted mean value of the two numbers  $x_1$  and  $x_2$ , where the sum of the weights  $w_1$  and  $w_2$  is required to be one:  $w_1 + w_2 = 1$ . The mediant  $\frac{a+c}{b+d}$  of  $\frac{a}{b}$  and  $\frac{c}{d}$  can be thought of as a weighted mean value, as

$$\frac{a+c}{b+d} = \frac{b}{b+d}\frac{a}{b} + \frac{d}{b+d}\frac{c}{d},$$

i.e, the weights  $w_1 = \frac{b}{b+d}$  and  $w_2 = \frac{d}{b+d}$  are determined by the denominators only. We have similarly for n numbers

$$\frac{a_1}{b_1} \oplus \dots \oplus \frac{a_n}{b_n} = \frac{b_1}{b_1 + \dots + b_n} \frac{a_1}{b_1} + \dots + \frac{b_n}{b_1 + \dots + b_n} \frac{a_n}{b_n}.$$

We will use this mean value property in Section 3 (Lemma 17). Of course, when considering weighted mean values, the weights  $w_1, ..., w_n$  are usually considered to be independent of  $x_1, ..., x_n$ . The above remark has a significance as a way of more exactly specify where  $\frac{a+c}{b+d}$  is positioned relatively to  $\frac{a}{b}$  and  $\frac{c}{d}$ . Note also that this mean value is not a well-defined mean value of rational numbers, since  $\frac{da_1}{db_1} \oplus \frac{a_2}{b_2} \neq \frac{a_1}{b_1} \oplus \frac{a_2}{b_2}$  in general. It is rather a mean value of pairs of numbers. We remark that the mediant operation  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1+a_2}{b_1+b_2}$  is isomorfic to the

vector addition in linear algebra:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2).$$

This connection is further discussed in Section 4.2.

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#### $\mathbf{2.3}$ The Stern-Brocot tree

#### 2.3.1**Stern-Brocot** sequences

The Stern-Brocot tree is generated by starting with the sequence  $\frac{0}{1}, \frac{1}{0}$ . Iteratively, longer sequences are generated by inserting mediants in all intermediate spaces. Hence, the first Stern-Brocot sequences are

$$S_{0} = \left(\frac{0}{1}, \frac{1}{0}\right)$$

$$S_{1} = \left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$$

$$S_{2} = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right)$$

$$S_{3} = \left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right)$$

$$S_{4} = \left(\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{1}{0}\right)$$

The following figure shows the common way to depict Stern-Brocot tree in the literature, here including up to the fifth generation.



Fig. 1 The Stern-Brocot tree -  $S_5$ .

Since the mediant is a weighted mean value, the numbers are distinct, and all sequences  $S_n$  are in increasing order.

#### 2.3.2 Generations

If we omit numbers that are already generated, we can talk about generations of numbers, which will be important in this paper. The first generations are then the following:

$$G_{0} = \left(\frac{0}{1}, \frac{1}{0}\right)$$

$$G_{1} = \left(\frac{1}{1}\right)$$

$$G_{2} = \left(\frac{1}{2}, \frac{2}{1}\right)$$

$$G_{3} = \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}\right)$$

$$G_{4} = \left(\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, \frac{4}{1}\right).$$

Clearly,  $S_n = \bigcup_{i=0}^n G_i$  and  $G_n = S_n \setminus S_{n-1}$ . We denote the generation number of a ratio  $\frac{a}{b}$  by  $g(\frac{a}{b})$ , i.e.  $\frac{a}{b} \in G_{g(\frac{a}{b})}$ . Each positive rational number has a unique generation number, this follows from that each rational number occur exactly once in the tree (Theorem 2). It is obvious that  $|G_n| = 2^{n-1}$  except that  $|G_0| = 2$ , and that  $|S_n| = 2^n + 1$ .

Each number is a mediant of two numbers. These two numbers may by the terminology of graph theory be called *parents*. Since every second number in a

Stern-Brocot sequence is generated in the last step, and all other numbers in earlier steps, each number has one parent that belong to the previous generation and another that belongs to an earlier generation. The number  $\frac{1}{1}$  is the only exception to this. We call a parent in the previous generation the *close parent*, and the other parent the *distant parent*.

### 2.3.3 The tree and the graph

When depicting the Stern-Brocot tree in the literature, it is a tradition to denote the tree in a simplified and somewhat incorrect way. Edges to close parents are represented only. When disregarding the other edges, the Stern-Brocot tree is a binary tree, except for the generation consisting of  $\frac{0}{1}$  and  $\frac{1}{0}$ . When taking both kinds of edges into account, the graph is not a tree, if it is regarded as an undirected graph. For the results in this paper we need both kinds of edges. In the following figure the distant parent-offspring edges are marked with a dotted line.



Fig 2 The Stern-Brocot graph up to fifth generation  $-S_5 = \bigcup_{i=0}^5 G_i$ .

When we expicitely need both kinds of edges, we will talk about *the Stern-Brocot graph*. We then regard the graph as a directed graph where the edges are directed from lower to higher generations. As a directed graph it is a tree since there are no cycles. We use the term Stern-Brocot graph to emphasize that both kinds of egdes are equally important.

In the following argument, we exempt the nodes  $\frac{0}{1}$  and  $\frac{1}{0}$ . Considering close edges only, each node has one parent and two offsprings. Considering all edges, each node has two parents and an infinite number of offsprings, two in each higher generation. The graph is infinitely large, alternatively sufficiently large.

We furthermore remark that the size order among the entries from left to right is always preserved. Geometrically this means that the graph is a planar graph – the branches do not cross. The branches do not even shadow each other if we imagine the sun above the tree positioned in zenit.

#### 2.3.4 Stern-Brocot pairs

Nodes that are co-parents, i.e. has a common offspring, play an important role in this paper. A pair of rational numbers that are parents to  $\frac{a}{b}$  is called the *Stern-Brocot pair of*  $\frac{a}{b}$ , and is denoted by  $SB(\frac{a}{b})$ . By the construction, each ratio  $\frac{a}{b}$  has a unique pair of parents. Note that  $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = SB(\frac{a}{b})$  implies  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a}{b}$ , but the converse implication is usually false. For example,  $(\frac{2}{5}, \frac{3}{7})$ is the Stern-Brocot pair of  $\frac{5}{12}$ , but  $(\frac{1}{5}, \frac{4}{7})$  is not, although  $\frac{1}{5} \oplus \frac{4}{7} = \frac{5}{12}$ . We write the pair in size order, so if  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  is a Stern-Brocot pair we know that  $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ .

The significance of Stern-Brocot pairs is that it provides a decomposition where the numbers are as close as possible to the decomposed number. Note that if  $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$  is a Stern-Brocot pair there are no ratios in lowest terms in the interval  $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$  that have denominator smaller than  $b_1 + b_2$ . Except for the Stern-Brocot pair  $\left(\frac{0}{1}, \frac{1}{0}\right)$ , the two members of a Stern-Brocot pair always belong to different generations. Each Stern-Brocot pair  $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$  defines an infinite branch in the tree by repeated mediant-addition of the element that belongs to the lower generation. If  $g\left(\frac{a_1}{b_1}\right) > g\left(\frac{a_2}{b_2}\right)$  the branch runs to the right immediately below the element  $\frac{a_2}{b_2}$ :

$$B_R(\frac{a_2}{b_2}) = \{\frac{a_1 + na_2}{b_1 + nb_2}, n = 0, 1, 2, \dots\},\$$

while if  $g(\frac{a_1}{b_1}) < g(\frac{a_2}{b_2})$  it goes to the left below the ratio  $\frac{a_1}{b_1}$ :

$$B_L(\frac{a_1}{b_1}) = \{\frac{na_1 + a_2}{nb_1 + b_2}, n = 0, 1, 2, \dots\},\$$

Note that  $B_R(\frac{a_2}{b_2})$  goes to the right but appears to the left of  $\frac{a_2}{b_2}$ , and analogously for  $B_L$  (see Fig 3). For example, the first two branches in the tree are  $B_L(\frac{0}{1})$ and  $B_L(\frac{1}{0})$ , where  $B_L(\frac{0}{1})$  consist of all ratios where the numerator is 1,  $\{\frac{1}{n}, n \in \mathbf{N}\}$ , while  $B_R(\frac{1}{0})$  is the set of natural numbers  $\mathbf{N} = \{1, 2, 3, ...\}$ . The next two branches are  $B_R(\frac{1}{1}) = \{\frac{n}{n+1}, n \in \mathbf{N}\}$  and  $B_L(\frac{1}{1}) = \{\frac{n+1}{n}, n \in \mathbf{N}\}$ .

Let us consider a certain node in the tree as a distant parent. Then all close co-parents to that node are located in the two branches below. Thus, the set of all co-parents to  $\frac{a}{b}$  in higher generations is  $C(\frac{a}{b}) = B_L(\frac{a}{b}) \cup B_R(\frac{a}{b})$ . Of course, the two branches never intersect.



Fig 3 Branches  $B_L(\frac{1}{2})$  and  $B_R(\frac{1}{2})$  below  $\frac{1}{2}$ .

## 2.3.5 Anchestor sequences

We use the term *ancestor sequence*  $A(\frac{a}{b})$  to a ratio  $\frac{a}{b}$  for the sequence containing all parents, parent's parents, and so on, in size order. The ratio itself is included in the anchestor sequence. We only include one of the anchestors  $\frac{0}{1}$  and  $\frac{1}{0}$ . If  $\frac{a}{b} \leq 1$  the anchestor  $\frac{0}{1}$  is included, otherwise  $\frac{1}{0}$  is included. It follows from the construction that the anchestor sequence contains exactly one member for each generation from 0 to  $g(\frac{a}{b})$ . The anchestor sequence  $A(\frac{a}{b})$  is a subsequence to  $S(g(\frac{a}{b}))$ , and forms a conelike set in the Stern-Brocot tree. For example, the anchestor sequence of  $\frac{2}{5}$  is  $A(\frac{2}{5}) = (\frac{0}{1}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{1}{1})$ .



Fig 4  $A(\frac{5}{8})$  – anchestor set of  $\frac{5}{8}$  inside double lines.

As we shall see, the numbers in  $A(\frac{a}{b})$  are the numbers that occur in the sequences  $\begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} a \\ b \end{bmatrix}$ . The floor sequence consists of  $\frac{a}{b}$  and the numbers in  $A(\frac{a}{b})$  to the left of  $\frac{a}{b}$ , while the ceiling sequence consists of  $\frac{a}{b}$  and those to the right. In Section 3 we prove this and specify exactly how c is related to the members in the set  $A(\frac{a}{b})$ .

We sometimes use the term anchestor set, instead of anchestor sequence, if the order is irrelevant

#### 2.3.6 Proofs of basic properties of the Stern-Brocot tree

We next establish fundamental properties of the Stern-Brocot tree. The proofs follow those in [4].

**Theorem 2** Each non-negative rational number occurs exactly once in the Stern-Brocot tree, and in lowest terms.

*Proof*: No number can occur twice or more, since a second occurence in a Stern-Brocot sequence would violate the strict increasing order of numbers.

We next show that all numbers in the tree are in lowest terms. Of course, the ratio  $\frac{l}{L}$  is in lowest terms if there are integers a and b so that la + Lb = 1. We show by induction that we have Lr - lR = 1 if  $\frac{l}{L}$ ,  $\frac{r}{R}$  are adjacent numbers in any Stern-Brocot sequence. From this it follows that both  $\frac{l}{L}$  and  $\frac{r}{R}$  are in lowest terms.

It is clear that the pair  $\frac{0}{1}, \frac{1}{0}$  fulfil the condition Lr - lR = 1. For the induction it is enough to show that if we have Lr - lR = 1, then L(l+r) - l(L+R) = 1is also true. This is a trivial calculation.

Finally we prove that any ratio  $\frac{a}{b} \ge 0$ , where a and b are relatively prime, appears in the Stern-Brocot tree. For this we use the natural binary search algorithm in the tree, starting with  $(\frac{0}{1}, \frac{1}{0})$ , where we in each step pick the interval  $(\frac{l}{L}, \frac{l+r}{L+R})$  or  $(\frac{l+r}{L+R}, \frac{r}{R})$  that contains  $\frac{a}{b}$ . A third possibility is  $\frac{l+r}{L+R} = \frac{a}{b}$ , in which case we have found the appearance of  $\frac{a}{b}$  in the tree. We need to show that this third case necessarily happens at some point during the search algorithm.

We will show that from the inequalities  $\frac{l}{L} < \frac{a}{b} < \frac{r}{R}$  and Lr - lR = 1 it follows that  $l + L + r + R \le a + b$ . Since at least one of the numbers l, L, rand R increase by at least one in each step of the iteration, while a and b are constants, the inequalities  $\frac{l}{L} < \frac{a}{b} < \frac{r}{R}$  cannot be valid for an infinite number of steps. Hence, the third case  $\frac{l+r}{L+R} = \frac{a}{b}$  necessarily happens. In proving  $l + L + r + R \le a + b$ , we start by noting that the inequalities  $\frac{l}{L} < \frac{a}{L} < \frac{r}{R}$  give aL = bL > 1 and bR = rR > 1.

 $\frac{l}{L} < \frac{a}{b} < \frac{r}{R}$  give  $aL - bl \ge 1$  and  $br - aR \ge 1$ . If these inequalities are multiplied by r + R respective l + L we get

$$(aL - bl)(r + R) \ge r + R,$$
  
$$(br - aR)(l + L) \ge l + L.$$

Addition of the inequalities and cancellation to the left gives

$$a(Lr - Rl) + b(-lr + rL) \ge l + L + r + R,$$

so from Lr - lR = 1 we get

$$a+b \ge l+L+r+R.$$

The theorem is proved.

In order to describe how rapidly the ratios grows in the tree, it is well known that the denominators of the numbers  $x \in G_n$ , x < 1 are at least n and at most  $F_n$ . Here  $F_n$  is the *n*:th member in the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ...,defined iteratively by  $F_{n+2} = F_{n+1} + F_n$  and  $F_0 = 1, F_1 = 1$ .

Furthermore, the Stern-Brocot tree gives the best possible rational approximations of an irrational number. We may extend the definition of an anchestor set to an irrational number q, by iteratively picking an interval  $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)$  to which q belongs as in the proofs of Theorem 2, giving an infinite anchestor set A(q). The set contains the best rational approximations of q with limited denominator:

**Theorem 3** Suppose that q > 0 is irrational. If  $\frac{a}{b} \notin A(q)$  and  $q < \frac{a}{b}$ , then there is a ratio  $\frac{c}{d} \in A(q)$  so that  $d \le b$  and  $q < \frac{c}{d} < \frac{a}{b}$ . Similarly, if  $\frac{a}{b} \notin A(q)$  and  $q > \frac{a}{b}$ , then there is a ratio  $\frac{c}{d} \in A(q)$  so that  $d \le b$  and  $\frac{a}{b} < \frac{c}{d} < q$ .

For a proof, see [4].

For example, the anchestor set of the golden mean  $\phi = (\sqrt{5} + 1)/2 \approx$ 1.61803... is  $A(\phi) = \{\frac{1}{0}\} \cup \{F_n/F_{n-1}, n \in \mathbf{N}\}$ . Thus, ratios of Fibonacci numbers give best possible rational approximations of  $\phi$ . We remark that  $F_n$  can explicitly be calculated using  $\phi$  as  $F_n = (\phi^n - (-\phi)^{-n}/\sqrt{5})$ .

We conclude this section by contributing to the knowledge about the Stern-Brocot graph with a graph theoretic observation. We here study the Stern-Brocot graph and not the tree – both kinds of edges are important. This graph is a directed graph where each edge has a direction from a parent to its offspring. A path from  $\frac{a_0}{b_0}$  to  $\frac{a_n}{b_n}$  is a sequence of nodes  $(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{n-1}}{b_{n-1}}, \frac{a_n}{b_n})$ , where  $\frac{a_{k-1}}{b_{k-1}}$ is a parent to  $\frac{a_k}{b_k}$  for all k = 1, ..., n.

The observation says that the number of paths from a ratio to the closest first anchestor,  $\frac{0}{1}$  or  $\frac{1}{0}$ , simply is given by the denominator of the ratio.

**Theorem 4** For a ratio  $\frac{a}{b} \leq 1$ , the number of distinct paths in the Stern-Brocot graph from  $\frac{0}{1}$  to  $\frac{a}{b}$  is b. For a ratio  $\frac{a}{b} > 1$ , the number of distinct paths from  $\frac{1}{0}$ to  $\frac{a}{b}$  is b.

*Proof*: We prove this by induction. Suppose that  $\frac{a}{b} \leq 1$ . The induction starts with the observations that  $\frac{1}{1}$  has denominator 1 and one single path to  $\frac{0}{1}$ , which takes care of the case  $\frac{a}{b} = 1$ , and that  $\frac{1}{2}$  has denominator 1 and one single path to  $\frac{1}{1}$ , which takes care of the case  $\frac{a}{b} = 1$ , and that  $\frac{1}{2}$  has denominator 2 and two paths to  $\frac{0}{1}$ . For  $\frac{1}{2}$ , there is one path directly from  $\frac{0}{1}$ , the path  $(\frac{0}{1}, \frac{1}{2})$  and one via  $\frac{1}{1}$ , which is the path  $(\frac{0}{1}, \frac{1}{1}, \frac{1}{2})$ . For  $\frac{a}{b} < 1$ , let  $\frac{a_0}{b_0}$  be the close parent and  $\frac{a_1}{b_1}$  the distant parent. Thus,  $\frac{a}{b} = \frac{a_0 + a_1}{b_0 + b_1}$ . By the induction hypothesis,  $\frac{a_0}{b_0}$  has  $b_0$  paths to  $\frac{0}{1}$  and  $\frac{a_1}{b_1}$  has  $b_1$ paths to  $\frac{0}{1}$ .

paths to  $\frac{0}{1}$ 

Now, any path from  $\frac{a}{b}$  to  $\frac{0}{1}$  goes first to the close parent  $\frac{a_0}{b_0}$  or to the distant parent  $\frac{a_1}{b_1}$ . If the distant parent  $\frac{a_1}{b_1}$  is the first node,  $\frac{a_0}{b_0}$  do not belong to the path. Hence, the set of paths starting with  $\frac{a_0}{b_0}$  is disjoint to the set of paths starting with  $\frac{a_1}{b_1}$ . It follows that the total number of paths from  $\frac{a}{b}$  to  $\frac{0}{1}$  is  $b_0 + b_1$ . Since this is the denominator of  $\frac{a}{b}$ , the theorem is proven for  $\frac{a}{b} \leq 1$ . The proof in the case  $\frac{a}{b} > 1$  is very similar.

#### 3 Main results

#### Fundamental properties of $\begin{bmatrix} a \\ b \end{bmatrix}_c$ and $\begin{bmatrix} a \\ b \end{bmatrix}_c$ 3.1

We start by proving that for fixed a and b,  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  is decreasing and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  is increasing as functions of c.

**Lemma 5** For fixed a and b, we have for all c = 1, ..., b - 1 the inequalities  $\begin{bmatrix} a \\ b \end{bmatrix}_c \ge \begin{bmatrix} a \\ b \end{bmatrix}_{c+1}$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c \le \begin{bmatrix} a \\ b \end{bmatrix}_c$ .

*Proof*: We prove  $\begin{vmatrix} a \\ b \end{vmatrix}_c \ge \begin{vmatrix} a \\ b \end{vmatrix}_{c+1}$ .

For any (c+1)-decomposition  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}, \frac{a_{c+1}}{b_{c+1}}$ , the *c*-decomposition  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c} \oplus \frac{a_{c+1}}{b_{c+1}}$  has the property

$$\min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c} \oplus \frac{a_{c+1}}{b_{c+1}}) \ge \min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}, \frac{a_{c+1}}{b_{c+1}})$$

since  $\min(\frac{a_c}{b_c}, \frac{a_{c+1}}{b_{c+1}}) \leq \frac{a_c}{b_c} \oplus \frac{a_{c+1}}{b_{c+1}}$ . Perhaps it is possible to find other *c*-decompositions that increase the left side even more. Thus, for an optimal (c+1)-decomposition  $\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}, \frac{a_{c+1}}{b_{c+1}}$ , we have

$$\begin{bmatrix} a \\ b \end{bmatrix}_c \ge \min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c} \oplus \frac{a_{c+1}}{b_{c+1}}) \ge \min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}, \frac{a_{c+1}}{b_{c+1}}) = \begin{bmatrix} a \\ b \end{bmatrix}_{c+1}.$$

By an analogus argument for  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ , the lemma is proved.

We next consider the two cases of ratios  $\frac{a}{b}$  that are not represented in the Stern-Brocot tree: negative ratios and ratios which are not in lowest terms.

#### 3.1.1 Positive numerators are enough

By the next lemma it is enough to consider a = 0, ..., b - 1.

**Lemma 6** For all  $a \in \mathbf{Z}$  and  $b \in \mathbf{Z}_+$ , we have

$$\begin{bmatrix} a \\ b \end{bmatrix}_{c} = \lfloor \frac{a}{b} \rfloor + \begin{bmatrix} a \mod b \\ b \end{bmatrix}_{c},$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_{c} = \lfloor \frac{a}{b} \rfloor + \begin{bmatrix} a \mod b \\ b \end{bmatrix}_{c}, \text{ and }$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_{c} = \begin{bmatrix} a \mod b \\ b \end{bmatrix}_{c}$$

*Proof*: Consider a decomposition  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}$ . If we insert  $a_i = \lfloor \frac{a}{b} \rfloor b_i + r_i$ , i = 1, ..., c, into the minimum  $\min(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c})$ , we obtain

$$\min(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}) = \min(\frac{\lfloor \frac{a}{b} \rfloor b_1 + r_1}{b_1}, \dots, \frac{\lfloor \frac{a}{b} \rfloor b_c + r_c}{b_c})$$
$$= \lfloor \frac{a}{b} \rfloor + \min(\frac{r_1}{b_1}, \dots, \frac{r_c}{b_c}).$$

By summing the relations  $a_i = \lfloor \frac{a}{b} \rfloor b_i + r_i, i = 1, ..., c$ , it also follows that

$$\sum_{i=1}^{c} r_i = a \mod b.$$

Thus, for each decomposition  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}$ , there is a decomposition  $\frac{r_1}{b_1}, ..., \frac{r_c}{b_c}$  with  $\sum_{i=1}^c r_i = a \mod b$  and where  $\min(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}) = \lfloor \frac{a}{b} \rfloor + \min(\frac{r_1}{b_1}, ..., \frac{r_c}{b_c})$ . The two first equalities follow by maximizing or minimizing among the decompositions.

The third equality

$$\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \mod b \\ b \end{bmatrix}$$

follows immediately from the first two equalities and  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_c - \begin{bmatrix} a \\ b \end{bmatrix}_c$ . The lemma is proven.

This lemma seems to unravel a certain asymmetry between the floor and ceiling functions, since the floor function  $\lfloor \frac{a}{b} \rfloor$  occurs in both the floor and ceiling ratio statements. This is however superficial, and follows from the preference to use positive numbers if possible. Perhaps  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} \frac{a}{b} \end{bmatrix} + \begin{bmatrix} a \mod b - b \\ b \end{bmatrix}_c$  is a more appropriate counterpart to  $\lfloor \frac{a}{b} \rfloor_c = \lfloor \frac{a}{b} \rfloor + \lfloor \frac{a \mod b}{b} \rfloor_c$ .

In fact, the difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  has an extra symmetry, from which if follows that only at most  $\lceil b/2 \rceil$  values of a give distinct values.

**Lemma 7** For all  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}_+$ , we have

$$\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} b-a \\ b \end{bmatrix}_c = \begin{bmatrix} -a \\ b \end{bmatrix}.$$

*Proof*: Suppose that  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}$  is an optimal decomposition for  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  written in decreasing order, so that  $\max\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right) = \frac{a_1}{b_1}$  and  $\min\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right) = \frac{a_c}{b_c}$ . Then  $\max\left(-\frac{a_1}{b_1}, ..., -\frac{a_c}{b_c}\right) = -\frac{a_c}{b_c}$  and  $\min\left(-\frac{a_1}{b_1}, ..., -\frac{a_c}{b_c}\right) = -\frac{a_1}{b_1}$ , so

$$\begin{bmatrix} -a \\ b \end{bmatrix}_{c} = -\begin{bmatrix} a \\ b \end{bmatrix}_{c}$$
$$\begin{bmatrix} -a \\ b \end{bmatrix}_{c} = -\begin{bmatrix} a \\ b \end{bmatrix}_{c}$$

Hence,

$$\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} -a \\ b \end{bmatrix}_c,$$

and the lemma follows from  $\begin{bmatrix} -a \\ b \end{bmatrix}_c = \begin{bmatrix} b-a \\ b \end{bmatrix}_c$ , by Lemma 6.

It is furthermore enough to consider decompositions which ratios in the closed interval  $\left(\lfloor \frac{a}{b} \rfloor, \lceil \frac{a}{b} \rceil\right)$ .

**Lemma 8** There is a decomposition  $\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right)$  where  $\frac{a_i}{b_i} \in \left[\lfloor \frac{a}{b} \rfloor, \lceil \frac{a}{b} \rceil\right]$  for i = 1, ..., c where  $\max\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right) = \lceil \frac{a}{b} \rceil_c$ . There is also such a decomposition where  $\min\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right) = \lfloor \frac{a}{b} \rfloor_c$ .

*Proof*: Consider  $\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right)$ , where  $\frac{a_1}{b_1}$  is the smallest ratio and  $\frac{a_c}{b_c}$  is the largest, and that  $\frac{a_1}{b_1} < \lfloor \frac{a}{b} \rfloor$ . Then the decomposition  $\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right)$  can be replaced by  $\left(\frac{a_1+1}{b_1}, ..., \frac{a_c-1}{b_c}\right)$ , where the minimum cannot be smaller and the maximum cannot be larger. By repeating this argument, the lemma is proven.

#### **3.1.2** a/b not in lowest terms

We next take care of the case when a/b is not in lowest terms. First we single out the trivial cases. The most trivial case is when a is a multiple of b.

**Lemma 9** If  $\frac{a}{b} = n$  for some integer n, then  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_c = n$  for all  $1 \le c \le b$ .

*Proof*: Since a = nb, the decomposition  $\left(\frac{n(b-c+1)}{b-c+1}, \frac{n}{1}, ..., \frac{n}{1}\right)$  is best possible for any  $c \leq b$ .

In the next lemma, a and b may have a common divisor. We denote by GCD(a, b) the greatest common divisor of a and b.

**Lemma 10** Suppose that d = GCD(a, b). Then  $\begin{bmatrix} a \\ b \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_c = \frac{a}{b}$  if and only if  $c \leq d$ . Also,  $\begin{bmatrix} a \\ b \end{bmatrix}_c = 0$  if and only if  $c \leq GCD(a, b)$ .

*Proof*: Denote  $a = da_0$  and  $b = db_0$ . Here the decomposition  $\left(\frac{a_0(d-c+1)}{b_0(d-c+1)}, \frac{a_0}{b_0}, ..., \frac{a_0}{b_0}\right)$ , where all ratios are equal, is possible if and only if  $d - c + 1 \ge 1$ . The lemma follows.

Our final and exhaustive result when a and b have a common factor is the following.

**Lemma 11** Suppose that *d* is a positive integer. Then  $\begin{bmatrix} da \\ db \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_{\lceil c/d\rceil}$ ,  $\begin{bmatrix} da \\ db \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_{\lceil c/d\rceil}$  and  $\begin{bmatrix} da \\ db \end{bmatrix}_c = \begin{bmatrix} a \\ b \end{bmatrix}_{\lceil c/d\rceil}$ .

The lemma allows us to always consider ratios  $\frac{a}{b}$  in lowest terms. In this lemma the ceiling function in the index is genuin – it cannot naturally be replaced by a floor function. Lemma 11 is very natural if we consider a *c*-decomposition in *d* subsets, where we in each subset decompose the ratio  $\frac{a/d}{b/d}$  in parallel. There does not exist better decompositions than this, which follows by Lemma 17.

Next we generalize the abbreviation formula  $\frac{da}{db} = \frac{a}{b}$  to also include c:

Lemma 12 If a is an integer and b and c are positive integers we have

$$\begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}, \quad \begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}, \quad and \quad \begin{bmatrix} da \\ db \end{bmatrix}_{dc} = \begin{bmatrix} a \\ b \end{bmatrix}_{c}$$

The lemma follows by replacing c by dc in Lemma 11.

### **3.2** Connection to the Stern-Brocot tree

Next we relate the quantities  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ ,  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  to the Stern-Brocot tree. We consider the sequence of the two parents  $(\frac{l}{L}, \frac{r}{R})$  to  $\frac{a}{b}$ , i.e.  $(\frac{l}{L}, \frac{r}{R}) = SB(\frac{a}{b})$  as defined by the Stern-Brocot tree (see Section 2.3.4). We say that the replacement of  $\frac{a}{b}$  by  $SB(\frac{a}{b})$  is a partition  $\frac{a}{b}$ . A partition sequence  $P_c(\frac{a}{b})$  is a sequence written in increasing order consisting of c ratios. The sequence  $P_c(\frac{a}{b})$  is constructed from  $P_{c-1}(\frac{a}{b})$  by partitioning the occuring ratio that belongs to the latest generation. Since only ratios in the anchestor sequence of  $\frac{a}{b}$  appears, where all ratios belong to different generations, this procedure is well-defined. The process starts with  $P_1(\frac{a}{b}) = (\frac{a}{b})$ , followed by  $P_2(\frac{a}{b}) = (\frac{l}{L}, \frac{r}{R})$ . The next step depends on whether  $g(\frac{l}{L})$  or  $g(\frac{r}{R})$  is largest.

It is obvious that  $P_c(\frac{a}{b})$  contains c ratios and is a c-decomposition of  $\frac{a}{b}$ . We will find that it is a uniform decomposition (Lemma 17), in fact the unique such, and thus important for calculating  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ ,  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ , and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ . Considered as a set we have  $P_c(\frac{a}{b}) \subset A(\frac{a}{b})$ , but as sequences we may have

Considered as a set we have  $P_c(\frac{a}{b}) \subset A(\frac{a}{b})$ , but as sequences we may have  $c = |P_c(\frac{a}{b})| > |A(\frac{a}{b})| = g(\frac{a}{b}) + 1$  since  $P_c(\frac{a}{b})$  may have many repeated ratios, and possibly  $c > g(\frac{a}{b}) + 1$ . In fact,  $P_c(\frac{a}{b})$  has always very few distinct ratios.

#### **Lemma 13** $P_c(\frac{a}{b})$ contains at most three distinct ratios.

*Proof*: The ratio at the highest generation is partitioned into its parents, many times if it occurs repeatedly. This give rise to multiple versions of the two parents only, and no other ratios. Hence these three ratios are the only occuring distinct ratios. At the step when the last ratio is partitioned, there is exactly two distinct ratios. Only  $P_1(\frac{a}{b})$  contains one single distinct ratio. The lemma is proved.

Thus: the partition sequences are subsets of the anchestor set that usually have many repeated elements, builded by the algorithm from below in the tree and upwards. Our next aim is Theorem 16, that connects the Stern-Brocot tree to decompositions of  $\frac{a}{b}$ .

In Section 2.3.5 we defined a Stern-Brocot pair of a ratio  $\frac{a_0}{b_0}$ , denoted by  $SB(\frac{a_0}{b_0})$ , as a pair  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  related in that the two ratios  $\frac{a_1}{b_1}$  are  $\frac{a_2}{b_2}$  are the parents to  $\frac{a_0}{b_0}$  in the Stern-Brocot tree. We denote the greatest common divisor of a and b as GCD(a, b). Thus,

We denote the greatest common divisor of a and b as GCD(a, b). Thus, GCD(a, b) = 1 iff  $\frac{a}{b}$  is in lowest terms. Given a pair  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  with  $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_0}{b_0}$ , we next define the *Stern-Brocot operation*  $SBO(\frac{a_1}{b_1}, \frac{a_2}{b_2})$ . This operation maps the pair  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  onto another pair, which either is a Stern-Brocot pair or two equal ratios. It is defined as follows:

$$SBO(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = \begin{cases} SB(\frac{a_1}{b_1} \oplus \frac{a_2}{b_2}) \text{ if } GCD(a_0, b_0) = 1\\ (\frac{a_0}{b_0}, \frac{(d-1)a_0}{(d-1)b_0}) \text{ if } GCD(a_0, b_0) = d > 1 \end{cases}$$

We mentioned earlier that  $(\frac{2}{5}, \frac{3}{7})$  is a Stern-Brocot pair, but  $(\frac{1}{5}, \frac{4}{7})$  is not, although  $\frac{1}{5} \oplus \frac{4}{7} = \frac{5}{12}$ . The Stern-Brocot operation replaces  $(\frac{1}{5}, \frac{4}{7})$  by  $(\frac{2}{5}, \frac{3}{7})$ , so  $SBO(\frac{1}{5}, \frac{4}{7}) = (\frac{2}{5}, \frac{3}{7})$ .

Of course, the Stern-Brocot operation leaves Stern-Brocot pairs unchanged. Otherwise, the new pair is closer together. This is also the case if the ratio is not in lowest terms. This is the content of the following lemma, and the significance of the Stern-Brocot operation.

**Lemma 14** If  $\frac{a_1}{b_1} < \frac{a_2}{b_2}$  and  $SBO(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = (\frac{A_1}{B_1}, \frac{A_2}{B_2})$ , then either  $\frac{a_1}{b_1} < \frac{A_1}{B_1}$  and  $\frac{a_2}{b_2} > \frac{A_2}{B_2}$ , or  $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = (\frac{A_1}{B_1}, \frac{A_2}{B_2})$ .

*Proof*: In the case  $GCD(a_0, b_0) > 1$  the lemma is trivial. If  $GCD(a_0, b_0) = 1$ , we first remark that if  $a_1 = A_1$ , then also  $a_2 = A_2$  follows from  $a = a_1 + a_2 = A_1 + A_2$ , and similarly for the denominators. So if  $\frac{a_1}{b_1} = \frac{A_1}{B_1}$ , then also  $\frac{a_2}{b_2} = \frac{A_2}{B_2}$ . If  $GCD(a_0, b_0) = 1$ , both the inequalities  $\frac{a_1}{b_1} > \frac{A_1}{B_1}$  and  $\frac{a_2}{b_2} < \frac{A_2}{B_2}$  are impossible since by the construction of the Stern–Brocot tree there are no rational num-

bers in the interval  $(\frac{A_1}{B_1}, \frac{A_2}{B_2})$  except such that has denominator  $b_1 + b_2 = B_1 + B_2$ or larger. The lemma is proved.

We next use the Stern-Brocot operation to successively modify a decomposition D into D' by replacing a pair  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  in D by  $SBO(\frac{a_1}{b_1}, \frac{a_2}{b_2})$ . A decomposition where the Stern-Brocot operation has no effect on any possible pair in the decomposition is called an *invariant* decomposition.

**Lemma 15** Any c-decomposition  $D_c(\frac{a}{b}) = \left(\frac{a_1}{b_1}, \dots, \frac{a_c}{b_c}\right)$  of  $\frac{a}{b}$ , can in a finite number of steps be transformed into an invariant decomposition D' by applying the Stern-Brocot operation consecutively to all possible pairs of ratios. Then  $\min D \le \min D' \text{ and } \max D \ge \max D'.$ 

*Proof*: Any c-decomposition of  $\frac{a}{b}$  consists of ratios where the denominators are at most b - c + 1. If all numbers are multiplied with b!, we obtain integers only. Now we measure the variation of a decomposition D by the quantity V(D), which is defined as

$$V\left(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}\right) = \sum_{i=1}^c 2^{\left|\frac{a}{b} - \frac{a_i}{b_i}\right|b!}.$$

This measure of variation puts an absolute priority to minimizing the variations at large distances from  $\frac{a}{b}$ , i.e. where  $\left|\frac{a}{b} - \frac{a_i}{b_i}\right|$  is large. If a large distance decreases and all larger distances are unchanged, V will decrease, even if all smaller distances increases.

We have seen that the Stern-Brocot operation either has no effect or decreases the difference of the pair by moving both ratios closer. This means that if a pair  $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$  in a decomposition D is replaced by  $SB(\frac{a_1}{b_1}, \frac{a_2}{b_2})$ , giving a decomposition D', and if  $SB(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \neq (\frac{a_1}{b_1}, \frac{a_2}{b_2})$ , then V(D') < V(D). If the Stern-Brocot operation is iteratively applied to any starting decom-

position D, the same decomposition cannot reappear, since that would violate that V is decreasing. The number of decompositions is finite, from which if follows that the inevitable end result of the replacement process is an invariant decomposition. The lemma is proved.

In an invariant decomposition, all pairs are Stern-Brocot pairs, i.e. common parents to a certain ratio. The topology of the Stern-Brocot tree allows only very simple such decompositions.

**Theorem 16** If c > 1 there exist only invariant decompositions with two or three distinct elements.

The following figure depicts the two possible configurations, with two and with three elements.

Figure: Invariant Stern-Brocot decompositions of two and of three members.

*Proof*: If an invariant decomposition has two distinct elements only, it contains two elements that are parents to a third element, which is not part of the decomposition. Each element has one distant parent and one close parent. If we fix a distant parent A, which may appear in any location in the Stern-Brocot tree, as described in Section 2.3.4, all possible close co-parents to A are located in the two branches  $B_L(A)$  or  $B_R(A)$  below A. Any second parent B on these two branches form an invariant decomposition with two distinct elements. The common descendant of A and B is the next node on the same branch as B.

In the case of three distinct elements we have to consider how to add a third element C to the two existing parents A and B, so that C is a parent together with both A and B. If C is distant parent together with A, then it cannot be co-parent with B. Therefore, C has to belong to one of the two co-parent branches  $B_L(A)$  or  $B_R(A)$ . In order to be a co-parent also to B, there are only two possible locations for C: immediately above or immediately below B, on the same branch.

We next consider if an invariant decomposition is possible with four elements. We then try to add one fourth element D to the three existing, A, B and C. Again, if D is distant parent with A, then it cannot be co-parent with B or C. So the co-parenthood with A requires that D is located on one of the two branches below A. Similarly it has to be next to B or C on the same branch in order to co-parent with one of them. But then it will necessarily not be a co-parent with the remaining element on the same branch, C or B, to which it is not adjacent. Hence there is no invariant decomposition with four elements.

When considering invariant decompositions of five or more elements, we note that each subset of four element need to be an invariant decomposition in iteself. Since this is impossible, there does not exist an invariant decomposition of four or more distinct elements. The proof is complete.

Note that we have not yet ruled out the possibility that there may be several different invariant decompositions, one of which is uniform from above and one from below. By the next lemma there is only one invariant decomposition, and it is  $P_c(\frac{a}{b})$ , which is defined by a simple successive partition algorithm.

**Lemma 17**  $P_c(\frac{a}{b})$  is an invariant decomposition of  $\frac{a}{b}$ , and a unique uniform *c*-decomposition of  $\frac{a}{b}$ .

*Proof*:  $P_c(\frac{a}{b})$  is generated by successively partitioning the element of the highest generation. The two elements that result from such a partition are thus co-parents. The other two possible pairs are also co-parents. This follows from the fact that one Stern-Brocot sequence is generated from the previous by adding offsprings in the intermediate spaces between two elements, where one is an offspring of the other. Hence, in terms of the conventional graph terminology applied to the Sterm-Brocot tree, we have that an offspring and the close parent are always co-parents to another offspring.

It remains to prove that there can be no other invariant decomposition than  $P_c(\frac{a}{b})$  that also is a uniform decomposition. Suppose that  $P_c(\frac{a}{b})$  has the two distinct members  $\frac{a_1}{b_1}$  and  $\frac{a_2}{b_2}$  and  $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ . Then we can write  $\frac{a}{b}$  in two alternative ways, where the last one is in terms of weighted mean values:

$$\frac{a_1}{b_1} \oplus \dots \oplus \frac{a_1}{b_1} \oplus \dots \underbrace{a_2}_{b_2} \oplus \dots \oplus \underbrace{a_2}_{b_2} = c_1 \underbrace{b_1}_{c_1} \underbrace{a_1}_{b_1 + b_2} \underbrace{a_1}_{b_1} + c_2 \underbrace{b_2}_{b_1 + b_2} \underbrace{a_2}_{b_2} = \underbrace{a}_{b_1}$$

where  $c_1 + c_2 = c$ . Suppose furthermore that  $g(\frac{a_1}{b_1}) < g(\frac{a_2}{b_2})$ , so  $\frac{a_2}{b_2}$  is on a branch  $B_L(\frac{a_1}{b_1})$  below  $\frac{a_1}{b_1}$ . Can we move  $\frac{a_2}{b_2}$  on that branch and reach a different c-decomposition of  $\frac{a}{b}$ ? If we replace any of the  $\frac{a_2}{b_2}$ :s with ratios to the left on the same branch, we would adjust the mean value to the left. In order to maintain the mean value and keep a proper decomposition of  $\frac{a}{b}$ , we would be required to replace any or some of the  $c_1 \frac{a_1}{b_1}$ :s by any of the ratios on the branch. But all these changes would decrease the denominators of the ratios, so their sum cannot still be b. If we try to replace any of the  $\frac{a_2}{b_2}$ :s with ratios to the right on the same branch, we would adjust the mean value to the ratios. These changes would all increase the sum of denominators, so in this way we cannot find any proper c-decomposition.

By very similar arguments we may disprove the possibility of other invariant decompositions than  $P_c(\frac{a}{b})$  if  $\frac{a_1}{b_1} > \frac{a_2}{b_2}$  and if  $g(\frac{a_1}{b_1}) > g(\frac{a_2}{b_2})$ . Conservation of mean value, left-right, and denominator sum, up-down, also rules out any other invariant decomposition than  $P_c(\frac{a}{b})$  in the case that  $P_c(\frac{a}{b})$  consists of three distinct elements. If all ratios are moved to a different branch, either all denominators increase or decrease. Hence also such changes are impossible. The uniqueness of  $P_c(\frac{a}{b})$  as an invariant and uniform c-decomposition follows.

We remark the consequence that there always exist one decomposition that gives both the minimum and the maximum.

**Corollary 18** For any  $a \in \mathbb{Z}$  and  $b, c \in \mathbb{Z}_+$ , where  $c \leq b$ , there is a unique uniform decomposition, i.e. a decomposition  $\frac{a_1}{b_1}, ..., \frac{a_c}{b_c}$  so that  $\lfloor a \\ b \rfloor_c = \min(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c})$  and  $\lfloor a \\ b \rfloor_c = \max(\frac{a_1}{b_1}, ..., \frac{a_c}{b_c})$ .

We next geometrically construct the sequences  $\lfloor \frac{a}{b} \rfloor$  and  $\lceil \frac{a}{b} \rceil$  of length b, containing  $\lfloor \frac{a}{b} \rfloor_c$  and  $\lceil \frac{a}{b} \rceil_c$  for all  $c: 1 \leq c \leq b$ , respectively. To this end we split the anchestor sequence  $A(\frac{a}{b})$  in two sequences, the left and right anchestor sequences, denoted  $A_L(\frac{a}{b})$  and  $A_R(\frac{a}{b})$ . Define n as the position of  $\frac{a}{b}$  in  $A(\frac{a}{b})$ , i.e.  $A(\frac{a}{b})_n = \frac{a}{b}$ . Now we define  $A_L(\frac{a}{b})$  and  $A_R(\frac{a}{b})$  so that they contain the floor ratios and ceiling ratios respectively, and are ordered so that  $\frac{a}{b}$  is first in the sequence. Hence:  $A_L(\frac{a}{b})_i = A(\frac{a}{b})_{n-i+1}$  for i = 1, ..., n, and  $A_R(\frac{a}{b})_i = A(\frac{a}{b})_{n+i-1}$  for  $i = 1, ..., |A(\frac{a}{b})| - n + 1$ .

**Theorem 19** Suppose that  $\frac{a}{b}$  is in lowest terms and positive. Then the sequences  $\lfloor \frac{a}{b} \rfloor$  and  $\lfloor \frac{a}{b} \rceil$  are constructed successively by filling the sequences from left to right up to the length b by a number of copies of each entry of  $A_L(\frac{a}{b})$  and  $A_R(\frac{a}{b})$ , respectively, taken from the left. The first entry  $\frac{a}{b}$  is taken in one copy. The number of copies of any other entry equals the number of paths in the Stern-Brocot graph from that entry to  $\frac{a}{b}$ .

**Proof:** The set  $P_c(\frac{a}{b})$  contains only entries from  $A(\frac{a}{b})$ . We have to prove that the number of copies of an entry equals the number of paths from that entry to  $\frac{a}{b}$ . Instead we will prove a slightly different but equivalent statement. Note that the number of copies of an entry  $\frac{a_0}{b_0}$  in on of the sequences  $\lfloor \frac{a}{b} \rfloor$  or  $\lfloor \frac{a}{b} \rfloor$ equals the number of copies  $c_0$  of  $\frac{a_0}{b_0}$  that  $P_c(\frac{a}{b})$  contains for c so that  $P_{c+1}(\frac{a}{b})$ is the first where  $\frac{a_0}{b_0}$  is partitioned. This is so since in  $P_{c+1}(\frac{a}{b})$ , the ratio  $\frac{a_0}{b_0}$  is partitioned once, and then cannot be either floor or ceiling ratio. However the  $\frac{a_0}{b_0}$  is floor or ceiling ratio for all  $P_{c-i}(\frac{a}{b})$ ,  $i = 0, ..., c_0 - 1$ , since the partition that leads from  $P_{c-i}(\frac{a}{b})$  to  $P_{c-i+1}(\frac{a}{b})$  produces one more copy of  $\frac{a_0}{b_0}$ .

bo that leads from  $P_{c-i}(\frac{a}{b})$  to  $P_{c-i+1}(\frac{a}{b})$  produces one more copy of  $\frac{a_0}{b_0}$ . We next show by induction that the number of paths from  $\frac{a_0}{b_0}$  to  $\frac{a}{b}$  is  $c_0$ . We first prove the initial step of the induction. The close parent of  $\frac{a}{b}$  is immediately partitioned, so it has one copy both in the floor or ceiling sequence and in  $P_1(\frac{a}{b})$ . The distant parent is slightly more complicated. It may appear at any generation earlier than or equal to  $g(\frac{a}{b}) - 2$ . However, each partition of the other parent gives one more copy of the distant parent in both the floor or ceiling sequence and in  $P_i(\frac{a}{b})$ , until the distant parent is to be partitioned.

The general step of the induction is similar. Suppose that  $P_c(\frac{a}{b})$  has the two distinct members  $\frac{a_1}{b_1}, \frac{a_2}{b_2}$ , with  $c_1$  copies of  $\frac{a_1}{b_1}$  and  $c_2$  copies of  $\frac{a_2}{b_2}$ , and that  $\frac{a_1}{b_1} < \frac{a_2}{b_2}$  and  $g(\frac{a_1}{b_1}) < g(\frac{a_2}{b_2})$ . By the induction hypothesis, there exist  $c_1$  paths to  $\frac{a_1}{b_1}$  and  $c_2$  paths to  $\frac{a_2}{b_2}$ . Then  $\frac{a_2}{b_2}$  is partitioned  $c_2$  times giving rise to a new ratio  $\frac{a_3}{b_3}$ . The edge from  $\frac{a_2}{b_2}$  to  $\frac{a_3}{b_3}$  then allows  $c_2$  paths from  $\frac{a_3}{b_3}$  to  $\frac{a}{b}$ , since there is  $c_2$  paths from  $\frac{a_2}{b_2}$  to  $\frac{a}{b}$ . If  $g(\frac{a_3}{b_3}) > g(\frac{a_1}{b_1})$  we are done. If  $g(\frac{a_3}{b_3}) < g(\frac{a_1}{b_1})$ , partitions of  $\frac{a_1}{b_1}$  give  $c_1$  new copies of  $\frac{a_3}{b_3}$ . This is in accordance with that the edge from  $\frac{a_1}{b_1}$  to  $\frac{a_3}{b_3}$  allows  $c_1$  new paths from  $\frac{a_3}{b_3}$  to  $\frac{a}{b}$ . Since there are  $c_1$  paths from  $\frac{a_1}{b_1}$  to  $\frac{a}{b}$ . The partition of  $\frac{a_1}{b_1}$  gives a new ratio  $\frac{a_4}{b_4}$ . If  $g(\frac{a_3}{b_3}) > g(\frac{a_4}{b_4})$  we are done, otherwise we can proceed with a similar argument until we are done. The proof is complete.

A ratio  $\frac{a}{b}$  that may be negative or not in lowest terms can be handled with Theorem 19 by slight preparations.

**Theorem 20** For a general  $\frac{a}{b}$ , let d = GCD(a, b), and denote  $a_0 = a/d$  and  $b_0 = b/d$ . Then the sequences  $\begin{bmatrix} a_0 \mod b_0 \\ b_0 \end{bmatrix}$  and  $\begin{bmatrix} a_0 \mod b_0 \\ b_0 \end{bmatrix}$  are given by Theorem 19. Now the sequences  $\begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} a \\ b \end{bmatrix}$  can be constructed by adding  $\lfloor a_0/b_0 \rfloor$  to all entries and duplicating each entry of  $\begin{bmatrix} a_0 \mod b_0 \\ b_0 \end{bmatrix}$  and  $\begin{bmatrix} a_0 \mod b_0 \\ b_0 \end{bmatrix}$  and  $\begin{bmatrix} a_0 \mod b_0 \\ b_0 \end{bmatrix}$  respectively in d conies without changing their order. d copies without changing their order.

*Proof*: The theorem follows from Theorem 19, Lemma 6 and Lemma 11.

#### Algorithms for calculating $\begin{vmatrix} a \\ b \end{vmatrix}_c$ and $\begin{bmatrix} a \\ b \end{vmatrix}_c$ 3.3

We next give a self-contained algorithm for finding the values of  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ for any allowed combination of integers a, b and c. The following theorem does not require knowledge of the proofs in the paper.

In the following theorem the arrow  $\leftarrow$  is used as an assignment operator in the following three different ways:

1. If x is a number, " $x \leftarrow y$ " signifies assignment of the value of y to the variable x.

2. If x is a sequence, A or G, " $x \stackrel{p}{\leftarrow} y$ " means insertion of the value y between positions p and p+1 in the vector x. Thus, values to the right of p in the vector x are all juxtapositioned one step to the right, and the length of the vector x is incremented.

3. If x is a sequence, F or C, " $x \stackrel{(i)}{\leftarrow} y$ " means insertion of i copies of the value y after the last position in x. The length of the vector thus increases with i.

The algorithm works in two steps. First we construct the anchestor sequence  $A(\frac{a}{b})$  and the corresponding generation information  $G(\frac{a}{b})$ . In the second algorithm we work backwards in the anchestor sequence to construct the appropriate c-values and the sequences  $\begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} a \\ b \end{bmatrix}$ , i.e. the values of  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  and  $\begin{vmatrix} a \\ b \end{vmatrix}_c$  for all  $c: 1 \le c \le b$ .

**Theorem 21** If  $\frac{a}{b} = n$  for some integer n, then  $\lfloor \frac{a}{b} \rfloor_c = \lceil \frac{a}{b} \rceil_c = n$  for all  $1 \le c \le b$ . Otherwise, the sequences  $\lfloor \frac{a}{b} \rfloor$  and  $\lceil \frac{a}{b} \rceil$  are calculated by the following two algorithms.

Algorithm 1 – constructing  $A(\frac{a}{b})$ 

1. Let d = GCD(a, b). Assign  $a \leftarrow a/d$  and  $b \leftarrow b/d$ . Also assign  $a \leftarrow a/d$  $a \mod b$ .

2. Initialize:  $A = (\frac{0}{1}, \frac{1}{1}), G = (0, 1), l \leftarrow 0, L \leftarrow 1, r \leftarrow 1, R \leftarrow 1, g = 2,$ p = 2.

3. Do  $A \xleftarrow{p} \frac{l+r}{L+R}$ ,  $G \xleftarrow{p} g$  and  $g \leftarrow g+1$ , and go to the appropriate case 4a, 4b or 4c.

4a. If  $\frac{a}{b} = \frac{l+r}{L+R}$ , exit the iteration, i.e. go to 5. 4b. If  $\frac{a}{b} < \frac{l+r}{L+R}$ , do  $r \leftarrow l+r$  and  $R \leftarrow L+R$ , and go to 3. 4c. If  $\frac{a}{b} > \frac{l+r}{L+R}$ , do  $l \leftarrow l+r$ ,  $L \leftarrow L+R$  and  $p \leftarrow p+1$  and go to 3.

5. After the iteration, add  $\lfloor \frac{a}{b} \rfloor$  to all values in A. Now the maximal value in G is  $p = g(\frac{a}{b})$ .

Algorithm 2 – searching  $A(\frac{a}{b})$ 

1. The algorithm starts with values defined by Algorithm 1. Furthermore, initialize as follows: F and C are empty sequences,  $d = LGD(a, b), x \leftarrow p - 1$ ,  $F \stackrel{(d)}{\leftarrow} \frac{a}{b}, F \stackrel{(d)}{\leftarrow} A(x), y = p + 1, C \stackrel{(d)}{\leftarrow} \frac{a}{b}, C \stackrel{(d)}{\leftarrow} A(y), u \leftarrow d, v \leftarrow d, q \leftarrow p$ . 2.  $q \leftarrow q - 1$ . Pick i so that G(i) = q. 3a. If q = 0, exit the iteration, i.e. go to 4. 3b. If  $i < p, F \stackrel{(u)}{\leftarrow} A(x), x \leftarrow x - 1, v \leftarrow v + u$ . Go to 2. 3c. If  $i > p, C \stackrel{(u)}{\leftarrow} A(y), y \leftarrow y + 1, u \leftarrow v + u$ . Go to 2. 4.  $\lfloor \frac{a}{b} \rfloor = F, \lceil \frac{a}{b} \rceil = C$ . The complexity of the algorithm is O(b).

*Proof*: The algorithm is an implementation of Lemma 6, Theorem 19 and Theorem 20.

# 4 Further work and applications

Other than the computer science application described in Section 2, we here mention two further problem areas.

## 4.1 Number theory and practical applications

We here limit the discussion to applications of the ceiling-floor difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c$ . It can be given an fundamental application concerning the divisibility of numbers. It can be thought of not only answering whether two numbers are divisible, but also, if the answer is no, to quantify the distance to divisibility. Lemma 10 states that  $\begin{bmatrix} a \\ b \end{bmatrix}_c = 0$  if and only if  $c \leq GCD(a, b)$ , but if c > GCD(a, b), the number  $\begin{bmatrix} a \\ b \end{bmatrix}_c > 0$  is this quantification. For example, we remark that

$$\begin{bmatrix} i\\in+j \end{bmatrix}_i = \frac{1}{n(n+1)}$$

for all i = 2, 3, ... and for all  $1 \le j \le i - 1$ . This is not difficult to prove by considering optimal decompositions, or by studying the branch  $B_L(\frac{0}{1})$  in the Stern-Brocot tree.

An instance of the divisibility problem is the following: If a objects are to be distributed among b persons, each person obtains a/b objects in the mean. If a objects are to be distributed among b persons divided in c groups, then the ceiling-floor difference  $\begin{bmatrix} a \\ b \end{bmatrix}_c$  is the unaviodable unfairness. It is the mean amount given to each person in the most favoured group minus the mean amount for each person in the least favoured group, if the partition is made to minimize the unfairness.

One possible instance of this is if 14 persons obtain 11 baskets of fruit and cheese to divide among themselves after they have formed 6 groups. Another is 7 divers who need to return to the surface rapidly. They can distribute themselves among 7 submarines, and have in total 16 tubes of oxygen, which also need to be divided as fairly as possible. Here, by the above formula, the minimal unfairness is 1/6 gas tubes.

#### 4.2 Discrete linear algebra

The problem may be reformulated into a problem for integer vectors, i.e. vectors (a, b) where a and b are integers, and  $b \ge 1$ . The inverse problem can then be reformulated as a problem of decomposing the vector (a, b) into a sum of vectors  $(a_1, b_1), ..., (a_c, b_c)$  with positive y-components, where the directions  $a_i/b_i$  of the component vectors should be as unchanged as possible compared to the initial vector direction a/b. This formulation can naturally be generalized from 2 to n dimensions if we let the scalar product

$$\frac{a_1b_1 + \dots + a_nb_n}{\sqrt{a_1^2 + \dots + a_n^2}\sqrt{b_1^2 + \dots + b_n^2}}$$

measure the direction deviance between the integer vectors  $(a_1, ..., a_n)$  and  $(b_1, ..., b_n)$ , generalizing  $(a_1, a_2)$  and  $(b_1, b_2)$  or  $a_1/a_2$  and  $b_1/b_2$ . We may here assume that all integers are positive. Linear algebra with integer vectors have essential applications in computer geometry.

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ISSN 1101-1581 ISRN BTH-RES-02/06-SE

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